On the generation of surface waves by shear flows

By JOHN W. MILES
Department of Engineering, University of California, Los Angeles

(*Received* 27 *May* 1957)

SUMMARY

A mechanism for the generation of surface waves by a parallel shear flow $U(y)$ is developed on the basis of the inviscid Orr–Sommerfeld equation. It is found that the rate at which energy is transferred to a wave of speed c is proportional to the profile curvature $-U''(y)$ at that elevation where $U = c$. The result is applied to the generation of deep-water gravity waves by wind. An approximate solution to the boundary value problem is developed for a logarithmic profile and the corresponding spectral distribution of the energy transfer coefficient calculated as a function of wave speed. The minimum wind speed for the initiation of gravity waves against *laminar* dissipation in water having negligible mean motion is found to be roughly 100 cm/sec. A spectral mean value of the sheltering coefficient, as defined by Munk, is found to be in order-of-magnitude agreement with total wave drag measurements of Van Dorn. It is concluded that the model yields results in qualitative agreement with observation, but truly quantitative comparisons would require a more accurate solution of the boundary value problem and more precise data on wind profiles than are presently available. The results also may have application to the flutter of membranes and panels.

1. Introduction

The primary aim of the following analysis is the prediction of the energy transfer from a prescribed two-dimensional parallel shear flow in an inviscid incompressible fluid to a surface wave of prescribed wavelength and wave speed. The principal result to be obtained is that this transfer is proportional to the curvature of the velocity profile at that point in the profile where the mean air speed is equal to the wave speed. The formulation, as developed in § 3–5, is valid for surface waves of a rather general type and may be of some importance in the flutter of thin panels or membranes, but we shall focus our attention primarily on two-dimensional deep-water gravity waves.

We shall not attempt to develop the mechanism by which the surface wave energy is dissipated except in the initial formation of gravity waves on a previously smooth surface, and even in this case it will be necessary to introduce assumptions which would appear more questionable than those implied in the original model for energy transfer from the air to the surface wave.

Previous models of wave formation have been critically reviewed by Ursell (1956). The mathematical (as opposed to empirical) models may be briefly characterized as follows.

(1) *The Kelvin–Helmholtz model* (Lamb 1945, § 268) assumes a tangential discontinuity between uniform flows in air and water and predicts a critical wind speed of 650 cm/sec for the initiation of surface waves.

(2) *Jeffreys'* (1925) *sheltering model* (Lamb 1945, § 348) assumes a periodic component of wind stress in phase with the wave slope, which is supposed ot originate in the separation of the air flow over the wave crests and is described by a ' sheltering coefficient '. No method of calculating this coefficient is given, however, and the form of the results would be the same for other mechanisms.

(3) *The laminar flow models of Wuest* (1949) *and Lock* (1954) assume laminar flow in both air and water and yield the critical wind speed as in the corresponding analyses of stability for channel and boundary-layer flows.

(4) *Eckart's* (1953 b) *stochastic model* assumes an ensemble of gusts having prescribed durations in time and space. The details do not permit brief summary, but it should be noted that the assumed wind structure differs widely from that considered here. It may approximate a storm more closely, but it would be far more difficult to reproduce in a controlled experiment.

The model to be developed here improves on the Kelvin–Helmholtz model by allowing for distributed, rather than concentrated vorticity*, resembles Jeffreys' model in predicting an equivalent value for his sheltering coefficient, and is similar to the investigations of Wuest and Lock inasmuch as it leads to a boundary-layer stability problem. We define this model by the following assumptions.

(*a*) *The air.* We assume the air to be inviscid and incompressible and, in the absence of the wave motion, to have the prescribed mean shear flow $U(y)$. The subsequent disturbances in velocity and pressure associated with the wave motion will be assumed two-dimensional and sufficiently small to justify the linearization of the equations of motion. Turbulent fluctuations, albeit decisive in maintaining the mean shear flow, are neglected in the perturbation equations. The end result of these assumptions is the inviscid Orr–Sommerfeld equation (3.4).

(*b*) *The water.* We assume the motion of the water to be inviscid, incompressible, and irrotational, and the slope of the displaced surface to be small. Following Jeffreys (1925; Lamb 1945, § 348), the dissipation in the water will be calculated in the second approximation as a perturbation on the inviscid flow, assuming laminar

* The assumption of a concentrated vortex sheet in the Kelvin–Helmholtz model was criticized originally by Rayleigh (1880; also 1945, § 366).

motion and a free surface, but this result will appear only in the calculation of minimum wind speed (§ 6)*. We shall neglect any mean motion of the water that may be induced by the mean air flow. This neglect could be justified if either the actual mean velocity were small compared with the wave speed (so that it could only produce negligible inertial forces) or were laminar and confined to a layer thin compared with the wavelength.

(*c*) *Wave speed.* We assume that the access of inertia associated with the disturbance in the air has a negligible effect on the magnitude of the surface wave speed, only that component of the aerodynamic force that is in phase with the wave slope being regarded as important. This assumption, which is introduced only to simplify the algebraic operations by virtue of an expansion in the specific gravity of the air (and which is identical with that made in Jeffreys' sheltering theory), rules out the Kelvin–Helmholtz mechanism of wave formation *a priori*.

Referring to the assumption of an inviscid fluid, an appropriate Reynolds number for travelling waves in a slightly viscous fluid is

$$R = |U-c|/k\nu = |U-c|\lambda/2\pi\nu, \tag{1.1}$$

where U, c, λ, ν, and k denote the speed of the undisturbed flow, wave speed (phase velocity), wavelength, kinematic viscosity, and wave number. R may be evaluated for either air or water, but it will attain its smallest values in the air, both because of the higher kinematic viscosity and because $U-c$ may tend to zero. In the examples treated subsequently (§ 6, § 7) R will have values of the order of 10^3 at the outer edge of the boundary layer, though it tends to zero at the critical point where $U = c$. This implies that viscous forces in the air are likely to be important only in the neighbourhood of $U = c$, where their omission leads to a singularity in the equations of motion (§ 3). Previous studies of channel and boundary-layer flows (Lin 1955, ch. 5) suggest that this singularity provides an adequate representation of viscous effects (on *growing* waves) for sufficiently large average values of R. We specifically remark, on the other hand, that (in contrast to the Poiseuille or boundary-layer stability problems) the perturbation inertia forces for sufficiently large values of ck/ν may be expected to dominate the perturbation friction forces in the immediate neighbourhood of the boundary by virtue of the surface wave there. This implies, in particular, that the viscous drag forces of the air are negligible compared with the normal pressure in their effect on the surface wave (cf. preceding footnote).

* It might be thought that the air, behaving as in the boundary layer over an oscillating plane (cf. Lamb 1945, § 351) and having the higher kinematic viscosity, could contribute more damping than the water acting as a free surface (Lamb 1945, § 348). The ratio of the damping coefficients for small, laminar motions is found to be $(\mu_a/4\mu_w)[c^4/2g\nu_a\overline{(c-U)}]^{\frac{1}{2}}$, where c denotes wave speed, ν kinematic viscosity, μ true viscosity, and $\overline{c-U}$ an appropriate average through the region of unsteady viscous influence. This ratio is about 0·1 in the calculation of the minimum wind speed (§ 6).

The foregoing arguments appear sufficient to justify an inviscid model for the determination of a first approximation to the disturbed motion of the air and the consequent energy transfer to the wave, but they throw no light on the perturbation Reynolds stresses associated with the interaction between turbulent fluctuations in the original and perturbed flows (see Appendix). It is implicit in our model that these also are negligible compared with the perturbation inertia forces except in the neighbourhood of $U = c$. It would be difficult to provide an adequate *a priori* justification of this hypothesis, but it appears reasonable for gravity waves in consequence of their relatively long wavelength.

It perhaps may be regarded as obvious that, on the one hand, models such as we consider here represent a gross simplification of the real problem of generation of surface waves by wind and cannot be expected to yield quantitative results except under very special circumstances ; but that, on the other hand, our present very limited knowledge of turbulent flows makes substantial theoretical progress unlikely without appeal to simplified models. The ranges of validity of such models must be established primarily by comparison with experimental data; this is scarcely available in sufficiently detailed and reliable form to warrant firm conclusions at the present time, but it seems unlikely that any single model (for which mathematical analysis is feasible) will prove adequate for all circumstances*.

2. Equation of motion for surface waves

We consider first the equation governing the propagation of a small, but otherwise rather general type of surface wave that gives rise to an aerodynamic pressure p_a acting on the surface. Let m be the effective mass per unit area and L a linear operator such that $L\eta$ gives the stress resisting a deformation η of the surface; then the equation of motion reads

$$L\eta + m\eta_{tt} = -p_a. \tag{2.1}$$

We assume the surface wave to have the form

$$\eta(x, t) = ae^{ik(x-ct)}, \tag{2.2}$$

where a denotes the amplitude, k the wave number, c the phase velocity, x a coordinate measured in the direction of propagation, and t the time; here and subsequently, following the usual convention, the real parts of complex quantities are implied in the final interpretation. We also assume the aerodynamic pressure to be represented by

$$p_a = (\alpha + i\beta)\rho_a U_1^2 k\eta, \tag{2.3}$$

* " How does wind acting on water give rise to waves ? This is a question which has never been satisfactorily answered in detail. This is partly because we do not know the exact constitution of an ocean wave, nor of the wind, nor the exact way in which the one acts upon the other. Notwithstanding our ignorance of the details, the general nature of the operation has been fairly well made out." (Durand 1896). Ursell, in his (1956) survey of the problem, opens with the statement that " wind blowing over a water surface generates waves in the water by a physical process which cannot be regarded as known " and concludes that " the present state of our knowledge is profoundly unsatisfactory ".

where ρ_a denotes the density of the air, U_1 an as yet arbitrary reference speed for the air, $k\eta$ (in magnitude) the local slope of the wave, and $\alpha+i\beta$ a dimensionless pressure coefficient; in general, α and β are functions of both c and k which depend on the solution to the aerodynamic boundary value problem (§ 3). Substitution of (2.2) and (2.3) in (2.1) yields

$$L\eta - mk^2c^2\eta = -(\alpha+i\beta)\rho_a U_1^2 k\eta, \tag{2.4}$$

which is an eigenvalue equation relating c and k for the assumed wave motion.

The operator L may be eliminated from (2.4) by referring it to the *free surface wave speed* c_w in the absence of the aerodynamic pressure. In the latter event (2.4) reduces to

$$L\eta = mk^2c_w^2\eta, \tag{2.5}$$

which implies that $mk^2c_w^2$ is the eigenvalue of the operator L. Substituting (2.5) in (2.4), we may place the result in the form

$$c^2 = c_w^2 + s(\alpha+i\beta)U_1^2, \tag{2.6}$$

where s denotes the relative mass parameter

$$s = \rho_a/mk. \tag{2.7}$$

We emphasize that (2.6) does not represent an explicit solution for c inasmuch as α and β exhibit an implicit dependence thereon.

We turn now to the special case of gravity waves on deep water of density ρ_w. The effective mass then is ρ_w/k, while the operator L is simply $\rho_w g$, and we have (cf. Lamb 1945, § 228, § 229)

$$c_w^2 = g/k, \qquad s = \rho_a/\rho_w. \tag{2.8 a, b}$$

We remark that in the Kelvin–Helmholtz model (Lamb 1945, § 268) $\alpha+i\beta = -1$ if U_1 is chosen as $U-c$, where U is the (constant) air speed; in this approximation the aerodynamic pressure is directly proportional to, but 90° out of phase with, the wave slope and represents an access of inertia. In Jeffreys' model (Lamb 1945, § 348), on the other hand, only the component of aerodynamic pressure in phase with the wave slope is considered, and β corresponds to his sheltering coefficient if $U_1 = U-c$.

We shall attempt a solution of the eigenvalue equation (2.6) only for those values of c and k that render the magnitude of the third term small compared with c_w^2, on which hypothesis we may take $c = c_w$ as a zeroth approximation, evaluate $\alpha+i\beta$ with $c = c_w$, and take

$$c = c_w[1+\tfrac{1}{2}s(\alpha+i\beta)(U_1/c_w)^2] \tag{2.9}$$

as a first approximation. If we define the *negative damping ratio* ζ as the energy rate of growth *per radian**, the first approximation to ζ_a (*a* denoting air) is given by (now dropping the subscript w from c, as is permissible in this approximation)

$$\zeta_a = 2\mathscr{I}\{c\}/\mathscr{R}\{c\} = s\beta(U_1/c)^2. \tag{2.10}$$

Our primary problem, then, is the determination of β.

* The corresponding logarithmic decrement is $-\pi\zeta$, while the rate of growth of η *per unit time* is $\tfrac{1}{2}\zeta kc$.

The approximations (2.9) and (2.10) correspond to what we may designate as *weak coupling* of the two fluids. In the Kelvin–Helmholtz model, on the other hand, we have *strong coupling*, and at the transition from stability to instability [$U = U_m$, $s \ll 1$, see Lamb 1945, § 268 (4)]

$$s(U_m/c_m)^2 = 1, \tag{2.11}$$

where c_m (which depends on surface tension, as well as gravity) is the minimum value of c_w. We infer from this that the approximation (2.9) may be regarded as valid only if

$$|\alpha + i\beta|(U_1/U_m)^2 \ll (c/c_m)^2. \tag{2.12}$$

The fact that waves do begin to form at wind speeds very much less than the Kelvin–Helmholtz value U_m implies that the inequality (2.12) may be assumed in describing incipient wave formation. Of course, (2.6) could be solved without further approximation, but the boundary value problem to be solved (§ 3) would be far more difficult in consequence of the fact that the dependence of c on k therein could not be specified *a priori*.

3. Equations of motion for the air

The equations of motion governing a small perturbation of a two-dimensional shear flow $U(y)$ in an inviscid incompressible fluid of density ρ_a are (see Appendix)

$$\rho_a(u_t + Uu_x + vU_y) = -p_x, \tag{3.1 a}$$

$$\rho_a(v_t + Uv_x) = -p_y, \tag{3.1 b}$$

$$u_x + v_y = 0, \tag{3.1 c}$$

where u and v denote the x- and y-components of the perturbation velocity, p the perturbation pressure, and subscripts partial differentiation. Introducing a stream function according to

$$u = -\psi_x, \qquad v = \psi_y, \tag{3.2 a, b}$$

and assuming (by virtue of linearity) ψ and p to exhibit the same dependence on x and t as η in (2.2), we obtain

$$\rho_a[(U-c)\psi_y - U_y\psi] = p, \tag{3.3 a}$$

$$\rho_a k^2(U-c)\psi = p_y. \tag{3.3 b}$$

Elimination of p then yields

$$(U-c)\psi_{yy} - [k^2(U-c) + U_{yy}]\psi = 0. \tag{3.4}$$

Equation (3.4), which was studied originally by Rayleigh (1880; also 1945, § 366 (7)), is the inviscid form of the well known Orr–Sommerfeld equation (Lin 1955, (1.3.15)). We observe that the omission of the viscous terms leads to a (regular) singularity at $U = c$.

The subsequent analysis will be carried out in terms of the dimensionless variables ξ, w, and ϕ, defined by

$$\xi = ky, \qquad U - c = U_1 w(\xi), \qquad \psi = U_1\phi(\xi)\eta(x,t), \tag{3.5 a, b, c}$$

where U_1 is the arbitrary reference velocity introduced in (2.3). Introducing these in (3.4), we obtain

$$\phi'' - [1 + (w''/w)]\phi = 0. \tag{3.6}$$

The boundary conditions to be imposed on ϕ are dictated by the requirements that the interface (originally $y = y_0$) shall remain a streamline and that the disturbance shall die out at infinity. Noting that the horizontal velocity relative to the wave is approximately $U - c$, we obtain for the first condition

$$\psi_x/(U-c) = ik\eta \quad \text{at } y = y_0 + \eta \doteqdot y_0. \tag{3.7}$$

We remark that the approximation $u - c = U - c$ assumes only that the slope of the wave is small (amplitude small compared with wavelength, that is, $ka \ll 1$). If y were interpreted as the distance above a fixed plane it also would be necessary (in order to linearize the boundary condition) to assume the amplitude to be small compared with the length of any region in the neighbouring flow over which local variations must be considered. This would be a rather severe restriction in consequence of the large velocity gradients near the boundary, but it may be avoided by assuming y to specify the streamline that would have been a distance y above the undisturbed interface and applying the boundary condition at $y = y_0$. Perhaps the most satisfactory proof of this statement is to replace the independent variables x and y in (3.1 a, b, c) by x and Ψ, where Ψ is the total (as opposed to perturbation) stream function (von Mises transformation), write $U = U(\Psi)$ rather than $U(y)$, linearize with respect to the velocity perturbations about $U = U(\Psi)$, and apply the boundary condition (3.7) at $\Psi = \Psi_0$, where $U_0 = U(\Psi_0)$. The form of the equations is simpler in terms of y, however, and we shall retain it with the implicit understanding that it specifies a given streamline, rather than distance from a fixed plane. We also note that the boundary condition may be applied at any streamline, $y = y_0$, for which $ky_0 \ll 1$; thus, U_0 need not vanish, and it is not necessary to prescribe the shear profile all the way to the surface. In effect, we assume that a small transition layer, within which $0 \leqslant U \leqslant U_0$, moves with the surface wave*.

Introducing the dimensionless variables of (3.5), the condition (3.7) becomes

$$\phi_0 = w_0, \tag{3.8 a}$$

where the zero subscripts imply $\xi = \xi_0$. The requirement that the disturbance die out at infinity yields simply

$$\phi \to 0 \quad \text{as } \xi \to \infty. \tag{3.8 b}$$

The perturbation pressure is given by (3.3 a) or, in terms of the dimensionless variables, by

$$p = \rho_a U_1^2 k(w\phi' - w'\phi)\eta. \tag{3.9}$$

* The assumptions $ky_0 \ll 1$ and $|U-c| \gg v$ alone, without further linearization or the neglect of the dominant viscous forces, imply that p and v may be assumed constant across the transition layer within the approximations of Prandtl's boundary-layer theory, as applied to the *disturbed* flow.

Comparing (3.9) with (2.3) at $\xi = \xi_0$ and invoking (3.8 a) we obtain

$$\alpha + i\beta = w_0(\phi_0' - w_0'). \tag{3.10}$$

This completes the formulation of the aerodynamic boundary value problem, which now is stated by (3.6), (3.8 a, b), and (3.10).

4. Implicit solution

We next obtain an implicit integral expression for $\alpha + i\beta$, both in order to illuminate the decisive role of the singularity at $U = c$ ($w = 0$) and as the first step in an approximate determination of β. Multiplying both sides of (3.6) by ϕ^*, the complex conjugate of ϕ, integrating from $\xi = \xi_0$ to $\xi = \infty$, integrating $\phi''\phi^*$ by parts, and imposing the boundary conditions (3.8 a, b), we obtain

$$\int_{\xi_0}^{\infty} \{|\phi'|^2 + [1 + (w''/w)]\,|\phi|^2\}\, d\xi = [\phi^*\phi']_0^{\infty} = -w_0\phi_0'. \tag{4.1}$$

Now, from (3.10), since w is approximately real, β is given by the imaginary part of $w_0\phi_0'$, and the only contribution of the integral in (4.1) to this imaginary part must come from the singularity at $w = 0$. It follows that

$$\beta = -\mathscr{I}\left\{\int_{\xi_0}^{\infty} |\phi|^2(w''/w)\, d\xi\right\}. \tag{4.2}$$

The path of integration in (4.2) must be indented either over or under the singularity at $\xi = \xi_c$, where $w(\xi_c) = 0$, and on this choice depends the sign of β. This rather delicate and evidently decisive question has been discussed in some detail by Lin (1955, § 4.3 and ch. 8), and we merely state the conclusion that the path must be indented under the singularity*. Applying the calculus of residues then yields

$$\beta = -\pi|\phi_c|^2(w_c''/w_c'), \quad w_c = 0, \tag{4.3}$$

where the subscript c implies evaluation at $\xi = \xi_c$.

The result (4.3) implies that, *in the absence of dissipative forces*, a motion of the type (2.2) will be stable or unstable according as the curvature of the wind profile (U'') at that elevation where the wind speed is equal to the wave speed is positive or negative, respectively. We infer from this result that:

(1) only those waves having speeds in that range of the wind profile for which $-U''$ is large may be expected to grow; the lower limit for c may be imposed by the existence of a sub-layer of linear profile or by the interaction of the waves with the wind profile, while the upper limit will be rather less than the wind speed outside of the boundary layer;

(2) in the *initial* phases of wave formation, those waves having speeds well down in the profile (large $-U''$) may be expected to predominate;

* This assumes $c = c_w$ is real. In the next approximation $\mathscr{I}\{c\} > 0$, so that the singularity lies slightly above the real axis (assuming $w_c''/w_c < 0$), and the path of integration in (4.2) passes under the singularity without the necessity of indentation.

(3) experimental measurements of aerodynamic forces on stationary wave models (Stanton *et alia* 1932; Motzfield 1937; Thijsse 1951) may not yield significant values of such parameters as Jeffreys' sheltering coefficient, since the point at which $U = c$ then occurs right at the boundary.

We remark that the result (4.3) is directly related to Taylor's (1915) theorem that the momentum transfer from mean to disturbed flow in a shear profile is proportional to $-U''$; alternatively, the result may be interpreted in terms of the vortex theory of instability developed by Lin (1955, § 4.4), following earlier ideas of von Kármán (see also Rayleigh 1945, vol. 2, p. 391).

We now proceed to develop an integral expression for ϕ_c which may be used for an approximate determination of β in conjunction with (4.3). We start from (3.6) in the modified form

$$(w\phi' - w'\phi)' = w\phi. \tag{4.4}$$

Integrating both sides of this equation between $\xi = \xi_c$ and $\xi = \infty$, imposing the boundary condition (3.8 b) at $\xi = \infty$, and noting that

$$\lim_{\xi \to \xi_c} w\phi' = 0 \tag{4.5}$$

by virtue of the fact that ϕ' can have only a logarithmic singularity at the singular point of the differential equation*, we obtain

$$\phi_c = \frac{1}{w'_c} \int_{\xi_c}^{\infty} w\phi \, d\xi. \tag{4.6}$$

The principal attribute of (4.6), aside from the smoothing effect intrinsically associated with integration, is that it is locally insensitive to errors in (the approximation to) ϕ in the neighbourhood of the singular point $w = 0$.

We conclude this section by remarking that the approximations introduced up to this point may be regarded as inherent in our model. Additional approximations will be made in the following sections in order to simplify the mathematical analysis, but these are, at least in principle, subject to improvement without modification of the model.

5. Approximate solution for β

The exact solution of the differential equation (3.6) in terms of known functions for typical velocity profiles does not appear to be possible. The usual (Heisenberg) expansion in the parameter k^2 for the problem of thin (in wavelengths) boundary layers (Lin 1955, § 5.5) is not well suited to the relatively thick turbulent boundary-layer profiles that are of interest in the present problem.

* It may be proved that

$$\phi = \phi_c[1 + (w''_c w'_c)(\xi - \xi_c)\log(\xi - \xi_c) + O(\xi - \xi_c)]$$

in the neighbourhood of $\xi = \xi_c$ (cf. Rayleigh 1945, § 369; Lin 1955, (8.1.8)).

We shall attempt here only an approximate determination of β through (4.6), using the approximation

$$\phi = w(\xi)e^{-\xi}. \tag{5.1}$$

It is evident that (5.1) satisfies the boundary conditions (3.8 a, b), while the differential equation (3.6) is satisfied if either $|w''| \gg |w|$ (small ξ) or $|w''| \ll |w|$ (large ξ). These conditions are not, of course, sufficient to guarantee that the approximation (5.1) is uniformly valid for all ξ; in particular, it fails completely at the singular point ξ_c. We anticipate, nevertheless, that, in conjunction with (4.3) and (4.6), it will lead to an approximation to the magnitude of ϕ_c which is adequate for at least the qualitative description of the spectral distribution of the parameter β. We shall be content with this goal, remarking that the difficulties associated with the accurate determination of wind profiles scarcely justify anything more elaborate at this time.

Substituting (5.1) in (4.6) and that, in turn, in (4.3), we obtain,

$$\beta = -\pi \frac{w_c''}{w_c'^3}\left[\int_{\xi_c}^{\infty} e^{-\xi} w^2 \, d\xi\right]^2. \tag{5.2}$$

We shall evaluate the integral in (5.2) for a logarithmic profile, which, for turbulent flow over water, has the support of both theory (see Coles (1956) for a recent survey) and experiment (Roll 1948; see also Hay 1955; Ellison 1956). The form that most directly fits the description of § 3 above is

$$U(y) = U_0 + U_1 \log(y/y_0), \tag{5.3 a}$$

where U_0 is the lower bound of the logarithmic profile and U_1 the reference velocity, which is now defined as the coefficient of the *natural* logarithm in the velocity profile. The majority of experimental results are presented in the form (except that common logarithms are apt to be used)

$$U(y) = (U_*/\kappa)\log(y/z_0), \tag{5.3 b}$$

where $U_* = (\tau_0/\rho_a)^{1/2}$ is Prandtl's *shearing stress velocity*, τ_0 the shear stress at the surface, κ Kármán's universal *turbulence constant*, and z_0 an *effective roughness parameter*, in the notation of Roll (1948); the definitions of U_* and κ are those commonly adopted, but z_0 has been used in somewhat different senses by different writers. We remark that the reference pressure $\rho_a U_1^2$ now is proportional to the shearing stress τ_0.

The distribution of (5.3) evidently holds only for a limited range of U. We designate the wind speed at the outer edge of this range as U_∞; for our purposes, $U_\infty = 10U_1 = 25U_*$ is a rough but adequate approximation for the wind speed at two metres above the water*. The determination of the lower limit of validity, U_0, is both more important and more uncertain, reflecting the fact that, in reality, there is not a sharp transition from

* The reference position of two metres is that adopted by Roll; at 10 metres $U_\infty = 30U_*$ might be a better estimate.

logarithmic profile to sub-layer (laminar or otherwise). Estimates of both U_0 and z_0 will be discussed in § 7.

Substituting (5.3 a) or (5.3 b) in (3.5 b) and (5.2) and noting that, by definition,

$$c = U_0 + U_1 \log(y_c/y_0) = U_1 \log(y_c/z_0), \tag{5.4 a, b}$$

we obtain

$$w = \log(\xi/\xi_c), \tag{5.5}$$

$$\beta = \pi\xi_c\left[\int_{\xi_c}^{\infty} e^{-\xi}\log^2(\xi/\xi_c)\,d\xi\right]^2 \tag{5.6 a}$$

$$= \pi\xi_c^3\left[\int_1^{\infty} e^{-\xi_c u}\log^2 u\,du\right]^2. \tag{5.6 b}$$

The integral in (5.6 b) may be evaluated for small or moderate values of the argument ξ_c by writing the range of integration $(1, \infty)$ as the difference $(0, \infty)$–$(0, 1)$, identifying the infinite integral as the Laplace transform of $\log^2 u$ [Erdélyi 1954, § 4.6 (13)], and evaluating the finite

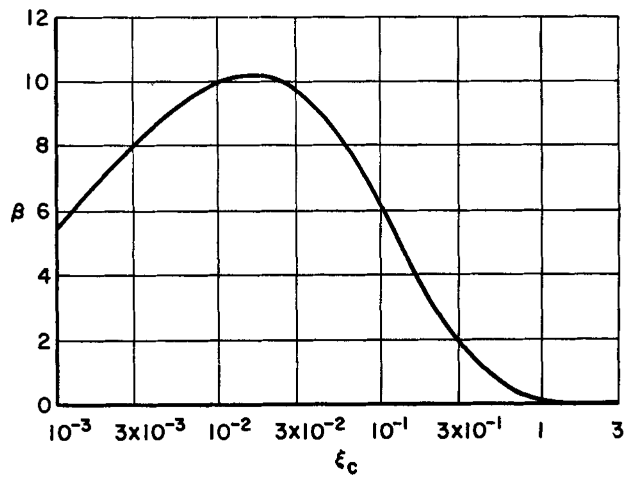

Figure 1. The function β (ky_c), as defined by (5.6) and (5.7)

integral as a power series, following the expansion of the exponential. An asymptotic development for large ξ_c may be obtained by the usual device of repeated integration by parts or by expanding $\log^2 u$ in powers of $(u-1)$. The end results of these operations are

$$\beta(\xi_c) = \pi\xi_c\left\{\tfrac{1}{6}\pi^2 + \log^2(\gamma\xi_c) + 2\sum_1^{\infty}\frac{(-)^n\xi_c^n}{n!\,n^2}\right\}^2, \tag{5.7 a}$$

$$\beta(\xi_c) \sim 4\pi\xi_c^{-3}e^{-2\xi_c}(1 - 6\xi_c^{-1} + 31\xi_c^{-2}\ldots), \tag{5.7 b}$$

where $\log\gamma \doteqdot 0{\cdot}5772$ is Euler's constant. We find that β has a maximum value of 10·2 at $\xi_c = 0{\cdot}017$ and that (5.7 a) is adequate for numerical calculation over the range of practical interest ($\xi_c < 1$). The asymptotic series (5.7 b), on the other hand, is accurate only for very large ξ_c and is of interest primarily as an indication of the exceedingly sharp cut-off of the damping ratio for very large values of ky_c.

The numerical values of β are plotted in figure 1.

6. Damping ratio *vs* wave speed

We now proceed to express β and ζ_a as functions of the dimensionless wave speed c/U_1. The corresponding wave number for gravity waves may be determined from (2.8 a), whence

$$k = g/c^2. \tag{6.1}$$

Solving (5.4 b) for y_c and multiplying by (6.1) then yields

$$\xi_c = ky_c = \Omega(U_1/c)^2 e^{c/U_1}, \tag{6.2}$$

where we have introduced the dimensionless, wind-profile parameter

$$\Omega = gz_0/U_1^2. \tag{6.3}$$

The function ky_c/Ω is plotted in figure 2. The corresponding values of β, as calculated from (5.7 a), and ζ_a/s, as given by (2.10), namely,

$$\zeta_a/s = (U_1/c)^2\beta[\Omega(U_1/c)^2 e^{c/U_1}], \tag{6.4}$$

are plotted in figures 3 and 4 for values of Ω that appear to be typical (cf. § 7).

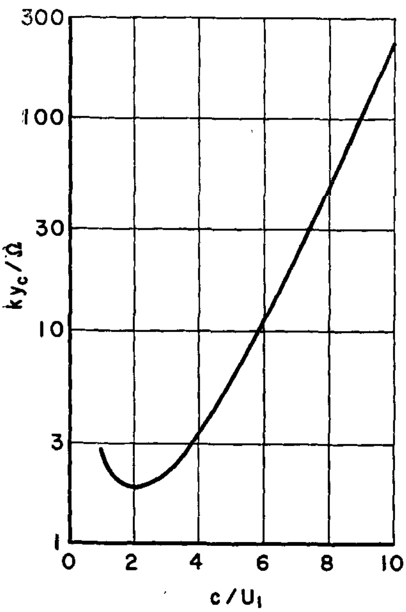

Figure 2. The variation of ky_c with wave speed; see (6.2) and (6.3)

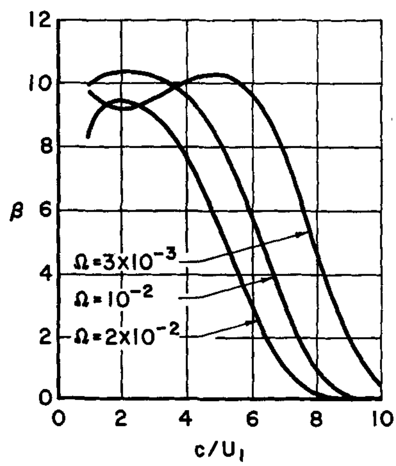

Figure 3. The parameter β, as determined by (5.7 a) and (6.2); cf. figures 1 and 2.

The damping ratio associated with viscous action in the water, based on the assumptions stated in § 1 (free surface with laminar flow), is given by (Lamb 1945, § 348)

$$\zeta_w = -4\nu_w k/c. \tag{6.5}$$

Eliminating k through (6.1) and superimposing ζ_a and ζ_w, we obtain

$$\zeta = \beta s U_1^2 c^{-2} - 4\nu_w g c^{-3}. \tag{6.6}$$

Equating this expression to zero yields

$$U_1 = (4\nu_w g/s)^{1/3}[(c/U_1)\beta]^{-1/3} \tag{6.7 a}$$

for the wind speed at which the energy available from the air stream is just sufficient to maintain a small amplitude gravity wave of speed c against laminar dissipation in the water. If we take the c.g.s. numerical values $\nu_w = 10^{-2}$, $g = 980$, and $s = 1{\cdot}2 \times 10^{-3}$, (6.7a) reduces to

$$U_1 = 32[(c/U_1)\beta]^{-1/3}\,\text{cm/sec} \quad \text{for } \zeta = 0. \tag{6.7b}$$

This result is plotted in figure 5 using the values of β given in figure 3.

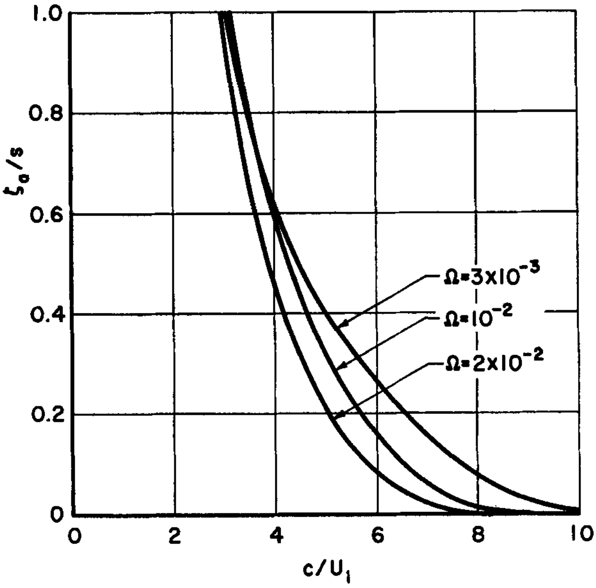

Figure 4. $\zeta_a s$, as given by (6.4).

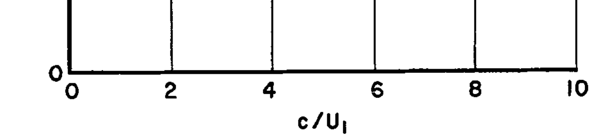

Figure 5. The minimum reference speed U_1 for the initiation of gravity waves, as given by (6.7b); the wind speed at two metres is roughly $10U_1$.

It appears from the results of figure 5 that the minimum wind speed (at two metres) for the initiation of *gravity* waves is of the order of magnitude of 100 cm/sec and might be as small as 80 cm/sec; the corresponding wave speeds would lie somewhere between 40–50 cm/sec. These figures are close to the respective measurements of Jeffreys (1924) and Scott Russell (1844), but it is important not to lose sight of the approximations on which they are based. The equivalent value of Jeffreys' sheltering coefficient, as given by

$$\beta_J = \beta U_1^2/(U_\infty - c)^2, \tag{6.8}$$

is roughly 0·3 (for $\Omega = 10^{-2}$, $c = 5U_1$, $U_\infty = 10U_1$).

7. The wind profile parameters

The available measurements on wind profiles over water (see Roll 1948; Neumann 1948; Hay 1955; also other papers cited by Ursell 1956 and Ellison 1956) all tend to confirm the logarithmic law posed in (5.3), but they provide little information on the laminar sub-layer and conflicting data on the effective roughness parameter. We consider here three possibilities.

(a) *Aerodynamically smooth flow*

If the departures from a plane surface do not penetrate the laminar sub-layer and the neglect of surface currents is justified the velocity profile should resemble that for flow near a smooth wall. Nikuradse's data yields (Prandtl 1952, p. 126)

$$U(y)/U_* = 5{\cdot}75\log_{10}(U_* y/\nu_a) + 5{\cdot}50, \tag{7.1}$$

corresponding to $\kappa = 0{\cdot}40$ and $z_0 = \nu_a/9{\cdot}06U_*$ in (5.3 b). We then have

$$z_0 = 1{\cdot}7\times 10^{-2}U_*^{-1}, \qquad \Omega = 2{\cdot}7U_*^{-3}, \tag{7.2 a, b}$$

in c.g.s. units ($g = 980$, $\nu_a = 0{\cdot}154$).

Roll has made measurements for wind blowing over smooth ponds which yield values of z_0 which are not only appreciably smaller than those predicted by (7.2 a) for $U_* < 10$ cm/sec but also decrease with U_*. The scatter in these data for small U_* is considerable, but the trend is quite definite. The most plausible explanation would appear to be in the existence of a surface current in the water, but further investigation of this point is desirable*.

If the logarithmic profile of (7.1) is matched in ordinate and slope to a linear profile it is found that $U_0 = 7{\cdot}8U_*$ at $U_* y_0/\nu_a = 1/\kappa$ (cf. Roll 1948). An examination of pipe data (Prandtl 1952, p. 128) reveals that the logarithmic and linear profiles are established only for $U/U_* > 15$ and $U/U_* < 5$, respectively, but the logarithmic profile does not appreciably overestimate U''/U' as a function of U for $U > 8U_*$.

(b) *Moderate wind over wavy water*

Roll found that for values of U_* between about 10 and 30 cm/sec and heights up to two metres above a disturbed surface his measurements (which exhibited a much smaller scatter than those for $U_* < 10$ cm/sec)

* Prandtl (1952, p. 130), referring to this data, states that " for values of U_* below about 10 cm/sec the flow in the lowest 2 metres seems to be laminar ". This interpretation is difficult to accept, since his own " universal velocity function " (*ibid*, p. 128) implies a value of y from 1–3 mm for transition at $U_* = 10$ cm/sec.

could be fitted to (7.1) when y was measured from the wave crests. In terms of the distance from the mean surface, he found

$$U/U_* = 5{\cdot}75 \log_{10}[U_*(y+a)/\nu_a] + 1{\cdot}85, \tag{7.3}$$

where a was the amplitude of the surface waves. This implies that

$$z_0 = 7{\cdot}3 \times 10^{-2} U_*^{-1}, \qquad \Omega = 11{\cdot}6 U_*^{-3}. \tag{7.4 a, b}$$

It should be recalled that we have defined y as a streamline parameter, so that the velocity profile of (7.3) is not directly significant and serves only to illustrate the magnitude of possible deviations from (7.1). We shall see (figure 6) that the resulting differences in ζ_a are negligible.

The results (7.3) and (7.4) are characteristic of aerodynamically smooth flow, so that it appears reasonable again to assume that (7.3) is valid down to about $U_0 = 8U_*$.

(*c*) *Fully developed rough flow*

It has been argued (cf. Ursell's (1956) and Ellison's (1956) remarks) from dimensional considerations that, since (i) the apparent roughness of the water surface depends on the waves produced by the wind and (ii) the roughness parameter in pipe flows is found to be independent of viscosity and to depend only on the nature of the roughness for $U_* z_0/\nu_a > 3$ (*rough flow*), z_0 should be proportional to U_*^2/g for sufficiently rough flow. We remark here that this conclusion appears to assume that capillary ripples are unimportant in determining the apparent roughness, although this would appear doubtful; nevertheless, Hay (1955) has made measurements for values of U_* between about 25 and 55 cm/sec that tend to confirm this hypothesis and lead to the approximate results (which easily might be off by as much as 30%; cf. Ellisons' remarks)

$$z_0 = 8 \times 10^{-5} U_*^2, \qquad \Omega = 1{\cdot}25 \times 10^{-2}. \tag{7.5 a, b}$$

The corresponding value of U_0 is probably best estimated as that at $y_0 = 30z_0$, namely $U_0 = 8{\cdot}5U_*$.

The discrepancy between (7.4) and (7.5) emphasizes the order of magnitude of the uncertainties with which we deal; for example, at $U_* = 30$ cm/sec, where the data of Roll and Hay overlap, the values of z_0 obtained from (7.4 a) and (7.5 a) are 2×10^{-3} cm and 7×10^{-2} cm, respectively. In any event, our model cannot be expected to have more than qualitative significance for rough flow.

The foregoing results are illustrated in figures 6 and 7, where we have plotted ζ_a *vs* λ $(= 2\pi/k)$ for the combinations of data indicated in table 1 (the values of Ω used in the actual calculations were rounded off to the parenthetic values in order to reduce the numerical work; the rounding errors are smaller than the other uncertainties). The differences in z_0 for given U_* are seen to be relatively unimportant. We again emphasize that the results for the higher wind speeds must be regarded as especially dubious in consequence of the linearization. The value of $-\zeta_w$ given by (6.5) also is plotted as a reference base, but the actual damping from the water almost certainly would be very much larger for values of U_* as high as 30 cm/sec. The values of $c/U_1 = 2, 3, 4$, and 5, corresponding to various, possible lower limits $U_0 = c$ are marked off; on the basis of the preceding

Item	U_* (cm/sec)	z_0 (cm)	U_{200} (cm/sec)	Ω	Eqs.
a_5	5	$3{\cdot}4\times10^{-3}$	110	$2{\cdot}2\times10^{-2}$ (2×10^{-2})	(7.1), (7.2)
a_{10}	10	$1{\cdot}7\times10^{-3}$	290	$2{\cdot}7\times10^{-3}$ (3×10^{-3})	(7.1), (7.2)
b_{10}	10	$7{\cdot}3\times10^{-3}$	250	$1{\cdot}2\times10^{-2}$ (10^{-2})	(7.3), (7.4)
b_{30}	30	$2{\cdot}4\times10^{-3}$	850	$4{\cdot}3\times10^{-3}$ (3×10^{-3})	(7.3), (7.4)
c_{30}	30	$7{\cdot}2\times10^{-2}$	600	$1{\cdot}2\times10^{-2}$ (10^{-2})	(7.5)

Table 1. The parenthetic values for Ω were used in calculating figure 6.

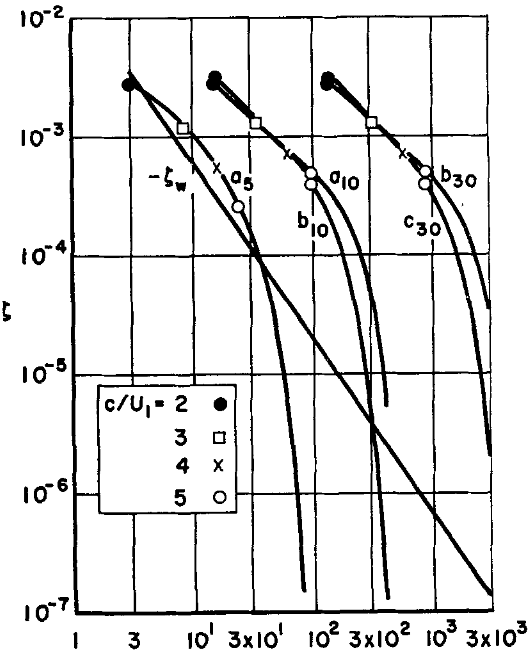

Figure 6. Damping *vs* wavelength for the data of table 1.

remarks, the logarithmic profile probably does not overestimate ζ_a for $c > 4U_1 (= 10U_*)$ but certainly would do so for values of c as low as $2U_1$ $(= 5U_*)$.

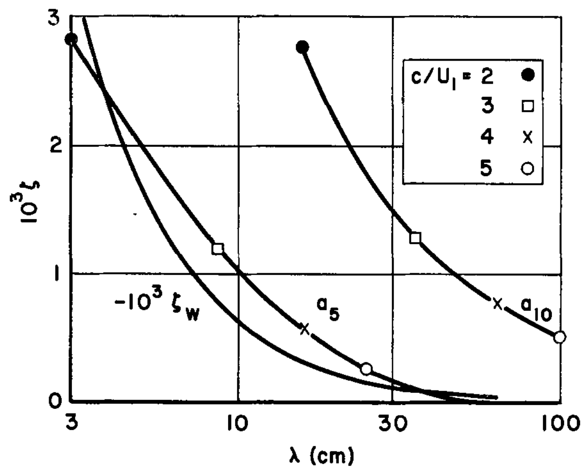

Figure 7. Linear plot of cases a_5 and a_{10}; cf. table 1 and figure 6.

8. Mean tangential wave drag

A rough check on the mean value of β (the minimum wind speed estimates of § 6 provide a check only on values of β in the neighbourhood of its maximum) may be obtained by calculating the (space or time) mean tangential drag on the waves, namely,

$$\tau_x = \overline{\mathscr{R}\{p_a(ik\eta)^*\}}. \tag{8.1}$$

Munk (1955), using data from Van Dorn's (1953) experiments, has obtained the expression

$$\tau_x = c'\rho_a(g\nu_w)^{-1/3}U_\infty^3, \tag{8.2}$$

where $c' = 0{\cdot}68 \times 10^{-6}$ if U_∞ is measured at 10 metres above the surface.

Let $S(k, \theta)$ be the power spectral density of the surface* in the notation of Eckart (1953 a), θ being the angle between wavefront normal and wind direction (so that the effective component of the wind is $U \cos\theta$); the power spectral density of the wave slope then is $k^2S(k, \theta)$, and substituting p_a in (8.1) from (2.3) yields

$$\tau_x = \rho_a U_1^2 \int_0^\infty \int_{-\pi}^{\pi} \beta \cos^2\theta k^2 Sk \, dk d\theta. \tag{8.3}$$

The spectrum for a fully developed sea (Neumann, as cited by Munk (1955)) may be approximated by

$$\left.\begin{aligned} k^2S(k,\theta)k\, dkd\theta &= 4(2\pi)^{-1/2}\theta_0^{-1}\sigma^2 e^{-2(c/U_\infty)^2} d(c/U_\infty)\, d\theta(|\theta| < \tfrac{1}{2}\theta_0), \\ &= 0 \quad (|\theta| > \tfrac{1}{2}\theta_0), \end{aligned}\right\} \tag{8.4}$$

where σ^2 is the mean square slope, which may be expressed in the form

$$\sigma^2 = b(g\nu_w)^{-1/3}U_\infty \tag{8.5}$$

* The surface displacement now is assumed to be composed of a distribution of sine waves having random phases.

with $b = 1{\cdot}1 \times 10^{-4}$ (Cox & Munk 1954). Munk (1955) also introduces a sheltering coefficient β_M (s in his notation), defined such that

$$c' = \beta_M \, bf(\theta_0), \tag{8.6}$$

where, for our model (in which $U \cos\theta$ would appear as the effective wind speed), $f(\theta_0)$ is simply the mean value of $\cos^2\theta$ over $(-\tfrac{1}{2}\theta_0, \tfrac{1}{2}\theta_0)$. Substituting (8.4) and (8.5) in (8.3) and comparing the result with (8.2) and (8.6), we find

$$\beta_M = \frac{4}{\sqrt{(2\pi)}}\left(\frac{U_1}{U_\infty}\right)^3 \int_0^{U_\infty/U_1} \beta e^{-2(c/U_\infty)^2} \, d\left(\frac{c}{U_1}\right). \tag{8.7}$$

Using the values of β given by figure 3 with $\Omega = 10^{-2}$ (the value most likely to be representative of a fully developed sea), assuming $\beta \equiv 0$ for $c < 3U_1$, and choosing $U_\infty = 12U_1$ (at 10 metres above the surface), we obtain $\beta_M = 2{\cdot}0 \times 10^{-2}$, which compares with the value 10^{-2} inferred by Munk from observed rates of wave growth. The corresponding value of c', using $b = 1{\cdot}1 \times 10^{-4}$, is $2{\cdot}1 \times 10^{-6}\, \overline{\cos^2\theta}$; with Munk's value of θ_0, namely 130°, we have $c' = 1{\cdot}5 \times 10^{-6}$, which compares with the experimental value of $0{\cdot}68 \times 10^{-6}$. We emphasize, however, that our estimate of β_M might be off by a factor of two either way in consequence of our estimates of Ω and c_0.

We conclude that the calculated value of c' is in agreement with the experimental value in at least order of magnitude, from which we infer that the mean value of β, weighted as in (8.7), also is of the right order of magnitude.

9. Conclusions

We conclude that the model of an inviscid shear flow in air over water having zero mean motion provides an explanation of the generation of gravity waves which is in qualitative agreement with observations of minimum wind speed and mean tangential drag. Quantitative comparisons would require, on the theoretical side, a more accurate solution of the boundary value problem, and, on the experimental side, much more precise data on wind profiles. We also emphasize that the model is intrinsically limited by the neglect of turbulent interaction between surface wave and wind profile and the linearization of the equations of motion. Finally, we note that the results of § 2–5, regarding energy transfer from shear flows to surface waves, are applicable to such problems as the flutter of membranes and panels.

I am grateful to Lester Lees of the California Institute of Technology, Nicholas Rott of Cornell University, Cark Eckart, Walter Munk and Charles Cox of the Scripps Institute of Oceanography, and my colleague Kurt Forster, for helpful discussion and criticism during the course of the above research.

Appendix

The Euler and continuity equations for a viscous incompressible fluid can be written

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=\frac{1}{\rho}\frac{\partial p_{ij}}{\partial x_j}, \tag{A1 a}$$

$$\frac{\partial u_i}{\partial x_i}=0, \tag{A1 b}$$

where x_i denotes a Cartesian coordinate, u_i a velocity component, p_{ij} a stress tensor component, ρ the fluid density, and the usual summation convention is implied. We resolve the dependent variables according to

$$u_i=U_i+u_i'+u_i'', \qquad p_{ij}=P_{ij}+p_{ij}'+p_{ij}'', \tag{A2 a, b}$$

where U_i+u_i' and $P_{ij}+p_{ij}'$ represent a solution to (A1 a, b) having the two-dimensional (x_1 and x_2) mean values (with respect to either x_3 or t) U_i and P_{ij} plus turbulent fluctuations u_i' and p_{ij}', and u_i'' and p_{ij}'' represent a small perturbation with respect to this solution. Substituting (A2 a, b) in (A1 a), neglecting terms of second order in the perturbation flow, and invoking the requirement that the unperturbed flow satisfy (A1 a), we obtain

$$\frac{\partial u_i''}{\partial t}+(U_j+u_j)\frac{\partial u_i''}{\partial x_j}+u_j''\frac{\partial(U_i+u_i')}{\partial x_j}=\frac{1}{\rho}\frac{\partial p_{ij}''}{\partial x_j}, \tag{A3 a}$$

$$\frac{\partial u_i''}{\partial x_i}=0. \tag{A3 b}$$

Taking mean values with respect to x_3, we may place the results in the form

$$\frac{\partial \overline{u_i''}}{\partial t}+U_j\frac{\partial \overline{u_i''}}{\partial x_j}+\overline{u_j''}\frac{\partial U_i}{\partial x_j}=\frac{1}{\rho}\frac{\partial}{\partial x_j}(\overline{p_{ij}''}-\overline{r_{ij}''}), \tag{A4 a}$$

$$\frac{\partial \overline{u_i''}}{\partial x_i}=0, \tag{A4 b}$$

where we have introduced the *perturbation Reynolds stress* (invoking the equations of continuity for both u_i' and u_i'' in its derivation)

$$\overline{r_{ij}''}=\rho(\overline{u_i'u_j''}+\overline{u_j'u_i''}) \tag{A5 a}$$

$$=\rho[\overline{u_i'(u_j''-\overline{u_j''})}+\overline{u_j'(u_i''-\overline{u_i''})}], \tag{A5 b}$$

with (A5 b) following from (A5 a) by virtue of $\overline{u_i'}=0$.

Equations (A4 a, b) are the equations of motion governing the two-dimensional perturbation of a two-dimensional shear flow if *two-dimensional* is understood to imply *mean value with respect to the transverse coordinate.* They differ from the equations of motion for a non-turbulent perturbation flow only in the presence of the $\overline{r_{ij}''}$, which represent the interaction between the fluctuations in the original and perturbation flows and, as might have been anticipated, are the first order perturbations of the usual Reynolds stresses $\overline{\rho u_i'u_j'}$.

Equations (3.1 a, b, c) follow from (A4 a, b) if we take $x_1=x$, $x_2=y$, $U_1=U(y)$, $U_2=U_3=0$, $\overline{u_1''}=u$, $\overline{u_2''}=v$, $\overline{p_{ij}''}=-\delta_{ij}p$, and $\overline{r_{ij}}=0$.

References

COLES, D. 1956 *J. Fluid Mech.* **1**, 191.

COX, C. & MUNK, W. 1954 *J. Mar. Res.* **13**, 198.

DURAND, W. F. 1896 *Sibley J. of Engineering*, April, 1896; reprinted in *Selected Papers of William Frederick Durand*, Durand Reprinting Committee, Calif. Inst. Tech., Pasadena, Calif. 1944.

ECKART, C. 1953 a *Phys. Rev.* **91**, 784.

ECKART, C. 1953 b *J. Appl. Phys.* **24**, 1485.

ELLISON, T. H. 1956 Article in *Surveys in Mechanics*. Cambridge University Press.

ERDÉLYI, A. (Ed.) 1954 *Tables of Integral Transforms*, Vol. 1. New York: McGraw-Hill.

HAY, J. S. 1955 *Quart. J. Roy. Met. Soc.* **81**, 307.

JEFFREYS, H. 1925 *Proc. Roy. Soc.* A, **107**, 189.

LAMB, H. 1945 *Hydrodynamics*. New York: Dover.

LIN, C. C. 1955 *The Theory of Hydrodynamic Stability*. Cambridge University Press.

LOCK, R. C. 1954 *Proc. Camb. Phil. Soc.* **50**, 105.

MOTZFIELD, H. 1937 *Z. angew. Math. Mech.* **17**, 193.

MUNK, W. 1955 *Quart. J. Roy. Met. Soc.* **81**, 320.

NEUMANN, G. 1948 *Z. Met.* **7/8**, 193.

PHILLIPS, O. M. 1957 *J. Fluid Mech.* **2**, 417.

PRANDTL, L. 1952 *The Essentials of Fluid Dynamics*. New York: Hafner.

RAYLEIGH, LORD 1880 *Proc. Lond. Math. Soc.* **11**, 57; *Scientific Papers* **1**, 474.

RAYLEIGH, LORD 1945 *The Theory of Sound*. New York: Dover.

ROLL, H. 1948 *Ann. Met., Hamburg*, **1**, 139.

SCOTT RUSSELL, J. 1844 *Rep. Brit. Ass.*, 317.

STANTON, T. E., MARSHALL, D. & HOUGHTON, R. 1932 *Proc. Roy. Soc.* A, **137**, 283.

TAYLOR, G. I. 1915 *Proc. Roy. Soc.* A, **215**, 1.

THIJSSE, J. T. 1951 *Nat. Bur. Stand., Wash., Circular* no. 521, 281.

URSELL, F. 1956 Article in *Surveys in Mechanics*. Cambridge University Press.

VAN DORN, W. 1953 *J. Mar. Res.* **12**, 249.

WUEST, W. 1949 *Z. angew. Math. Mech.* **29**, 239.

Note added at proof stage

A stochastic analysis recently presented by Phillips (1957) develops a model which appears more realistic than that proposed by Eckart (1936 b) and arrives at results in reasonable agreement with experiment. It seems that the mechanism proposed by Phillips is complementary to that presented here, but further investigation is required to establish the appropriate connections.